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QUASI-EXACTLY SOLVABLE POTENTIALS WITH TWO KNOWN EIGENSTATES

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Abstract

A new supersymmetry method for the generation of the quasi-exactly solvable (QES) potentials with two known eigenstates is proposed. Using this method we obtained new QES potentials for which we found in explicit form the energy levels and wave functions of the ground state and first excited state.

Key words: supersymmetry, quantum mechanics, quasi-exactly solvable potentials .

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1 Introduction

Quasi-exactly solvable (QES) potentials for which a finite number of eigenstates is explicitly known nowadays attract much attention. This is an intermediate class between the problems for which the spectrum can be found exactly and those which can not be solved. The first examples of QES potentials were given in [1–4]. Subsequently several methods were worked out for generating QES potentials and as a result many QES potentials were established [5–12]. One of the methods is the generation of new QES potentials using supersymmetric (SUSY) quantum mechanics [12–14]. This method applies the technique of SUSY quantum mechanics (see review [15]) to calculate the supersymmetric partner potential of the QES potential with $n+1$ known eigenstates. From the unbroken SUSY it follows that the supersymmetric partner is a new QES potential with n known eigenstates. It is worth stressing that the starting point of the SUSY method used in [12–14] for generating new QES potentials is the knowledge of the initial QES potentials. Note also that in [16, 17] using SUSY quantum mechanics a various families of conditionally exactly solvable (CES) potentials were constructed. The CES potentials are those for which the eigenvalues problem for the corresponding Hamiltonian is exactly solvable only when the potential parameters obey certain conditions [19].

In the present paper we propose a new SUSY technique for generating QES potentials with the two known eigenstates. In contrast to the previous paper [12–14] our SUSY method does not require the knowledge of the initial QES potential for the generation of new QES one. As a result, we obtained new QES potentials for which we found in explicit form the energy levels and wave functions of the ground and first excited states.

2 SUSY quantum mechanics and QES problems

Let us first take a look at the Witten's model of SUSY quantum mechanics. The SUSY partner Hamiltonians $H_\pm$ are given by

$$H_\pm = B^\mp B^\pm = -\frac{1}{2}\frac{d^2}{dx^2} + V_\pm(x), \tag{1}$$

where

$$B^\pm = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + W(x)\right), \tag{2}$$

$V_\pm(x)$ are the so-called SUSY partner potentials

$$V_\pm(x) = \frac{1}{2}\left(W^2(x) \pm W'(x)\right), \quad W'(x) = \frac{dW(x)}{dx}, \tag{3}$$

$W(x)$ is the superpotential.

Consider the equation for the energy spectrum

$$H_\pm\psi_n^\pm(x) = E_n^\pm\psi_n^\pm(x), \quad n = 0, 1, 2, ... \tag{4}$$

The Hamiltonians H_+ and H_- have the same energy spectrum except the zero energy ground state which exists in the case of the unbroken SUSY. Only one of the Hamiltonians $H_\pm$ has the zero energy eigenvalue. We shall use the convention that the zero energy eigenstate belongs to H_-. The corresponding wave function due to the factorization of the Hamiltonian H_- satisfies the equation $B^-\psi_0^-(x) = 0$ and reads

$$\psi_0^-(x) = C_0^- \ \exp\left(-\int W(x)dx\right), \tag{5}$$

C_0^- is the normalization constant. Here and below C denotes the normalization constant

of the corresponding wave function.

From the normalization condition it follows that

$$\mathrm{sign}(W(\pm\infty)) = \pm 1. \tag{6}$$

The eigenvalue and eigenfunction of the Hamiltonians H_+ and H_- are related by SUSY transformations

$$E^-_{n+1} = E^+_n, \quad E^-_0 = 0, \tag{7}$$

$$\psi^-_{n+1}(x) = \frac{1}{\sqrt{E^+_n}} B^+ \psi^+_n(x), \tag{8}$$

$$\psi^+_n(x) = \frac{1}{\sqrt{E^-_{n+1}}} B^- \psi^-_{n+1}(x). \tag{9}$$

The unbroken SUSY quantum mechanics, namely, the SUSY transformations are used for the exact calculation of the energy spectrum and wave functions (see review [15]). In the present paper we use SUSY quantum mechanics for the generation of the QES potentials with the two known eigenstates.

Suppose we study the Hamiltonian H_-, whose ground state is given by (5). Let us consider the SUSY partner of H_-, i.e. the Hamiltonian H_+. If we calculate the ground state of H_+ we immediately find the first excited state of H_- using the SUSY transformations (7), (8), (9). In order to calculate the ground state of H_+ let us rewrite it in the following form

$$H_+ = H^{(1)}_- + \epsilon = B^+_1 B^-_1 + \epsilon, \quad \epsilon > 0, \tag{10}$$

which leads to the following relation between potentials energies

$$V_+(x) = V^{(1)}_-(x) + \epsilon, \tag{11}$$

where $B_1^{\pm}$ are given by (2) with superpotential $W_1(x)$, similarly $V_-^{(1)}$ is given by (3) with $W_1(x)$, ϵ is the energy of the ground state of H_+ since $H_-^{(1)}$ has zero energy ground state.

Using this procedure N times we obtain N excited energy levels and corresponding wave functions of H_- in the following form

$$E_n^- = \sum_{i=0}^{n-1} \epsilon_i, \tag{12}$$

$$\psi_n^-(x) = C_n^- B_0^+ ... B_{n-2}^+ B_{n-1}^+ \exp\left(-\int W_n(x)dx\right), \tag{13}$$

where $n = 1, 2, ..., N$. In our notations $\epsilon_0 = \epsilon$, $B_0^{\pm} = B^{\pm}$, $W_0(x) = W(x)$. Operators $B_n^{\pm}$ are given by (2) with the superpotentials $W_n(x)$ which satisfy the set of equations

$$W_n^2(x) + W_n'(x) = W_{n+1}^2(x) - W_{n+1}'(x) + 2\epsilon_n, \quad n = 0, 1, ..., N-1. \tag{14}$$

This relations for superpotentials are the generalization of equation (11) for the case of N steps.

The formulas (12) and (13) can be considered as expressions for the energy levels and eigenfunctions for QES potential with N energy levels. But in order to calculate the eigenfunctions in explicit form it is necessary to solve the set of equations for superpotentials (14). Unfortunately in general, it is not possible to determine the superpotentials from (14) for arbitrary N.

Previously the set of equations for $W_n(x)$ was solved in the special cases of the so-called shape invariant potentials [18] and self-similar potentials [20, 21] and as a result many exactly solvable potentials were obtained [22] (see also review [15]).

We consider a more general case and do not restrict ourselves to the shape invariant or self-similar potentials. A *novelty* of this paper is that we are obtaining a general solution

of (14) for $N = 1$. Namely, the both superpotentials $W(x)$ and $W_1(x)$ in this case can be expressed via some function. As a result it is possible to construct general expression for QES potentials with explicitly known two eigenstates. It is the subject of the next section.

3 Constructing QES potentials with explicitly known two eigenstates

Let us consider set of equation (14) for $N = 1$. In this case we have one equation for two superpotentials $W(x)$ and $W_1(x)$

$$W^2(x) + W'(x) = W_1^2(x) - W_1'(x) + 2\epsilon. \tag{15}$$

Note, that (15) is the Riccati equation which can not be solved exactly with respect to $W(x)$ for a given $W_1(x)$ and vice versa. A new moment of this paper is that we can find such a pair of $W(x)$ and $W_1(x)$ that satisfies equation (15). For this purpose let us rewrite equation (15) in the following form

$$W_+'(x) = W_-(x)W_+(x) + 2\epsilon, \tag{16}$$

where

$$W_+(x) = W_1(x) + W(x), \tag{17}$$

$$W_-(x) = W_1(x) - W(x).$$

This new equation can be easy solved with respect to $W_-(x)$ for a given $W_+(x)$ and

vice versa. In this paper we use the solution of equation (16) with respect to $W_-(x)$

$$W_-(x) = (W_+'(x) - 2\epsilon)/W_+(x). \tag{18}$$

Then from (17) and (18) we obtain the pair of $W(x)$, $W_1(x)$ that satisfies equation (15)

$$W(x) = \frac{1}{2}\left(W_+(x) - (W_+'(x) - 2\epsilon)/W_+(x)\right), \tag{19}$$
$$W_1(x) = \frac{1}{2}\left(W_+(x) + (W_+'(x) - 2\epsilon)/W_+(x)\right),$$

here $W_+(x)$ is some function of x for which the superpotentials $W(x)$ and $W_1(x)$ given by (19) satisfy condition (6). As we see from (17) $W_+(x)$ satisfies the same condition (6) as $W(x)$ and $W_1(x)$ do.

Let us consider continuous function $W_+(x)$. Because $W_+(x)$ satisfies condition (6) the function $W_+(x)$ must have at least one zero. Then as we see from (18), (19) $W_-(x)$, $W(x)$ and $W_1(x)$ have the poles. In order to construct the superpotential free of singularities suppose that $W_+(x)$ has only one zero at $x = x_0$ with the following behaviour in the vicinity of x_0

$$W_+(x) = W_+'(x_0)(x - x_0). \tag{20}$$

In this case the pole of $W_-(x)$ and $W(x)$, $W_1(x)$ at $x = x_0$ can be cancelled by choosing

$$\epsilon = W_+'(x_0)/2. \tag{21}$$

Then the superpotentials free of singularities are

$$W(x) = \frac{1}{2}\left(W_+(x) - (W_+'(x) - W_+'(x_0))/W_+(x)\right), \tag{22}$$

$$W_1(x) = \frac{1}{2}\left(W_+(x) + (W_+'(x) - W_+'(x_0))/W_+(x)\right).$$

In the present paper we use this nonsingular solution for superpotentials in order to obtain nonsingular QES potentials. Substituting the obtained result for $W(x)$ (22) into (3) we obtain QES potential $V_-(x)$ with explicitly known wave function of ground state (5) and wave function of the excited state. The latter can be calculated using (13)

$$\psi_1^-(x) = C_1^- \; W_+(x) \exp\left(-\int W_1(x)dx\right). \tag{23}$$

As we see from (23) $\psi_1^-(x)$ has one node, because $W_+(x)$ has only one zero. Thus, $\psi_1^-(x)$ indeed is the wave function of first excited state.

Note, that all expressions depend on the function $W_+(x)$. We may choose various functions $W_+(x)$ and obtain in a result various QES potentials.

To illustrate the above described method we give four explicit examples of nonsingular QES potentials. First example is the well known QES potential and is specially chosen to show that our method works correctly. In the next examples we present new QES potentials which as far as we know have not been previously discussed in the literature.

3.1 Example 1

Let us first consider an explicit example which as we shall see gives the well known QES potential. Let us put

$$W_+(x) = A\left(\sinh(\alpha x) - \sinh(\alpha x_0)\right), \tag{24}$$

where $A > 0$, $\alpha > 0$. Then using (22) we obtain

$$W(x) = \frac{1}{2}\left(A\left(\sinh(\alpha x) - \sinh(\alpha x_0)\right) - \alpha\,\tanh\left(\frac{\alpha}{2}(x + x_0)\right)\right), \tag{25}$$

$$W_1(x) = \frac{1}{2}\left(A\left(\sinh(\alpha x) - \sinh(\alpha x_0)\right) + \alpha\,\tanh\left(\frac{\alpha}{2}(x + x_0)\right)\right),$$

which satisfy (6). Substituting this $W(x)$ into (3) we obtain the potential energy

$$V_-(x) = \frac{1}{2}\left(\frac{1}{4}A^2\left(\sinh(\alpha x) - \sinh(\alpha x_0)\right)^2 - A\alpha\cosh(\alpha x) + \frac{1}{2}A\alpha\cosh(\alpha x_0) + \frac{\alpha^2}{4}\right). \quad (26)$$

In the case of $\alpha > \frac{1}{2}A$ this is a non-symmetric double-well potential, x_0 is responsible for an asymmetry of potential. In the case of $x_0 = 0$ we obtain the symmetric QES potential. It is the special case of Razavy potential [3] with two known eigenstates. The QES potential (26) was derived in [9, 10] using the method elaborated in the quantum theory of spin systems (see also review [11]). It is interesting to note, that as we show, this potential has SUSY formulation.

We may calculate exactly the two eigenstates for potential (26). The distance between the ground energy level $E_0^- = 0$ and the first excited energy level E_1^- is

$$\epsilon = \frac{1}{2}\alpha A\cosh(\alpha x_0). \quad (27)$$

The wave function of the ground state can be easily calculated by (5)

$$\psi_0^-(x) = C_0^-\,\cosh\left(\frac{\alpha}{2}(x + x_0)\right)\exp\left(-\frac{A}{2\alpha}\cosh(\alpha x) + \frac{A}{2}\sinh(\alpha x_0)x\right). \quad (28)$$

For the wave function of the first excited state using (23) we obtain

$$\psi_1^-(x) = C_1^-\,\sinh\left(\frac{\alpha}{2}(x - x_0)\right)\exp\left(-\frac{A}{2\alpha}\cosh(\alpha x) + \frac{A}{2}\sinh(\alpha x_0)x\right). \quad (29)$$

The results (27), (28), (29) are the same as were obtained in [10] and we may claim that our SUSY method works correctly.

3.2 Example 2

Let us consider

$$W_+(x) = \frac{A\,\sinh(\alpha x)}{b + c\,\cosh(\alpha x)}, \tag{30}$$

which is a generalization of the function $W_+(x)$ considered in first example for $x_0 = 0$ and reproduces it in the special case $c = 0$. The parameters satisfy the condition $A > 0,\ \alpha > 0,\ c > 0$ and $b + c > 0$. The last condition ensures nonsingularity of $W_+(x)$.

The superpotentials $W(x)$ and $W_1(x)$ read

$$W(x) = \frac{1}{2}\left(\frac{(A + ac)\,\sinh(\alpha x)}{b + c\,\cosh(\alpha x)} - \frac{ab}{b + c}\,\tanh(\frac{\alpha x}{2})\right), \tag{31}$$

$$W_1(x) = \frac{1}{2}\left(\frac{(A - ac)\,\sinh(\alpha x)}{b + c\,\cosh(\alpha x)} + \frac{ab}{b + c}\,\tanh(\frac{\alpha x}{2})\right). \tag{32}$$

The superpotential $W(x)$ generates the following QES potential

$$V_-(x) = \frac{1}{8c^2}\left[\frac{(b^2 - c^2)(A + ac)(A + 3ac)}{(b + c\,\cosh(\alpha x))^2} - \frac{2b(A + ac)^2}{b + c\,\cosh(\alpha x)} + \frac{a^2bc^3}{(b + c)^2}\frac{1}{\cosh^2(\alpha x/2)} + \frac{(ac^2 + A(b + c))^2}{(b + c)^2}\right]. \tag{33}$$

The energy of the ground and first excited states are $E_0^- = 0$ and $E_1^- = \epsilon = aA/2(b+c)$ respectively.

The wave function of the ground and first excited states read

$$\psi_0^-(x) = C_0^-(\cosh(\alpha x/2))^{b/(b+c)}(b + c\,\cosh(\alpha x))^{-1/2 - A/2ac}, \tag{34}$$

$$\psi_1^-(x) = C_1^-\sinh(\alpha x)(\cosh(\alpha x/2))^{-b/(b+c)}(b + c\,\cosh(\alpha x))^{-1/2 - A/2ac}. \tag{35}$$

The wave functions of the ground state is square integrable for any parameters of su-

perpotential that satisfy the condition described above. The wave function of first excited state is square integrable when

$$\frac{c}{b+c} < \frac{A}{ac}. \tag{36}$$

In the opposite case the system has only localized ground state.

As far as we know the potential in general form (33) has not been previously discussed in the literature. This potential is interesting from that point of view that in special cases of parameters it reproduces the potentials studied early. Thus, in the limit $b \to 0$ the superpotential $W_+(x)$ (30) generates the well known exactly solvable Rosen-Morse potential. For $c \to 0$ one obtains the Razavy potential with two explicitly known eigenstates (Example 1).

It is interesting to consider the special case $A = ac$. In this case the first term in $W_1(x)$ drops up and $W_1(x)$ generates the SUSY partner $V_+(x)$ which is Rosen-Morse potential and can be solved exactly. Then using SUSY transformation (7) and (8) we can easily calculate the energy levels and wave functions of all states of Hamiltonian H_- with potential energy $V_-(x)$. In this special case $V_-(x)$ can be treated as CES potential and it corresponds to the one studied in [17].

3.3 Example 3

Consider the function $W_+(x)$ in the polynomial form

$$W_+(x) = ax + bx^3, \tag{37}$$

where $a > 0, b > 0$. The final result for QES potential is the following

$$V_-(x) = \frac{1}{8}(a^2 - 12b)x^2 + \frac{ab}{4}x^4 + \frac{b^2}{8}x^6 + \frac{3ab}{8(a+bx^2)^2} + \frac{3b}{8(a+bx^2)} - \frac{a}{4}. \tag{38}$$

The energy levels of the ground and first excited states are $E_0^- = 0$, $E_1^- = a/2$. Note, that two energy levels of this potential do not depend on the parameter b. The wave functions of those states read

$$\psi_0^-(x) = C_0^-(a + bx^2)^{3/4} e^{-x^2(2a+bx^2)/8}, \tag{39}$$

$$\psi_1^-(x) = C_1^- x(a + bx^2)^{1/4} e^{-x^2(2a+bx^2)/8}. \tag{40}$$

It is worth to stress that the case $b = 0$ corresponds to linear harmonic oscillator.

3.4 Example 4

Let us put

$$W_+(x) = \frac{Ax}{\sqrt{b^2 + x^2}}, \quad A > 0. \tag{41}$$

It is obviously that it is enough to consider only $b > 0$.

The QES potential in this case reads

$$V_-(x) = \frac{1 - A^2b^2}{8(b^2 + x^2)} - \frac{Ab^2}{2(b^2 + x^2)^{3/2}} - \frac{5b^2}{8(b^2 + x^2)^2} + \frac{(1 + Ab)^2}{8b^2}. \tag{42}$$

The energy levels of the ground and first excited states are $E_0^- = 0$, $E_1^- = a/2b$. The wave functions of those states read

$$\psi_0^-(x) = C_0^- \left(1 + \frac{b}{\sqrt{b^2 + x^2}}\right)^{1/2} e^{-\sqrt{b^2+x^2}(1+Ab)/2b}, \tag{43}$$

$$\psi_1^-(x) = C_1^- \frac{x}{\sqrt{b^2 + x^2}} \left(1 + \frac{b}{\sqrt{b^2 + x^2}}\right)^{-1/2} e^{-\sqrt{b^2+x^2}(-1+Ab)/2b}. \tag{44}$$

The wave function of first excited state is square integrable if $Ab > 1$. Otherwise only the localized ground state exists.

Thus as we see the proposed new SUSY method for constructing QES potentials gives us an opportunity to obtain new potentials for which two eigenstates are exactly known. In special cases our potentials reproduce those studied earlier.

4 Concluding remarks

In the present paper we propose a new SUSY method for constructing QES potentials with two explicitly known eigenstates. Namely, we obtain a general expression for QES potentials and wave functions of the ground and first excited states. This method is more general than those given in the literature before and in contrast to them does not require the knowledge of the initial QES potentials for constructing new QES potentials. But we must note that the proposed method is restricted to constructing QES potentials with explicitly known only two eigenstates. Naturally, there is a question about the generalization of this SUSY method for the case of QES potentials with explicitly known more than two eigenstates. In the case of three energy levels from (14) (where $N = 2$ corresponds to two excited levels) we obtain the set of two equations which relate three superpotentials. In order to obtain QES potential with three known eigenstates in explicit form we must solve this set of equations. It is more complicated problem than the case of one equation ($N = 1$) which was considered in the present paper. This problem will be a subject of a separate paper.

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